Computational Statistics and Data Analysis (MVComp2)

Exercise 4

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Semester Wi23/24
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1 Exponential distribution (2 points)

The exponential distribution is defined on the interval $[0, +\infty)$ as

$$p(x) \propto \exp(-x).$$

(a) Determine the moment-generating function $m(t)$ for the exponential distribution, $p(x)$.

(b) Verify that $\langle x^n \rangle = n!$.

2 Poisson distribution (2 points)

The Poisson distribution $p_\lambda(n)$ is given by

$$p_\lambda(n) = \frac{\lambda^n}{n!} \exp(-\lambda)$$

(a) Find the first three moments about the origin, $\langle n \rangle$, $\langle n^2 \rangle$, and $\langle n^3 \rangle$.[1]

(b) The Poisson distribution has the peculiar property that mean and variance are equal, implying

$$\frac{\langle n^2 \rangle}{\langle n \rangle^2} - 1 = \frac{1}{\langle n \rangle}.$$

Show that this is valid.

[1] A genuine attempt at solving the original assignment will also get you full points. For part (b) make sure to use the correct moments (the moments about the mean were discussed in class).

3 Generalized Fokker–Planck equation (4 points)

We propose to describe a continuous stochastic process of a probability density, where we will make use of a central-moment expansion and assume a Markovian (i.e., memoryless) description.

(a) **Formal-moment expansion**: Consider the (stationary[2]) probability distribution, $p(x)$, given its set of moments about the origin, μ'_n. Show that you can express the distribution in terms of its moments:

$$p(x) = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{\mu'_n}{n!} \delta(x).$$

Hint: The derivatives of the Dirac delta function follow the relation:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \, (ik)^n \mathrm{e}^{-ikx} = \left(-\frac{\partial}{\partial x}\right)^n \delta(x)$$

(b) **Kramers–Moyal expansion**: We convert p into a *transition probability function*, $p(x, t|x_0, t_0)$, where x and t denote the spatial coordinate and time, respectively. The moments about the origin become moments about the mean, i.e., we work with μ_n instead of μ'_n. Because of the expansion in (a), the time variable may only affect the moments, not the Dirac delta, i.e.,

$$p(x, t|x_0, t_0) = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{\delta(x - x_0)}{n!} \mu_n(t|x_0, t_0).$$

Invoke Markovianity of p by making use of the Chapman–Kolmogorov equation

$$p(x, t|x_0, t_0) = \int \mathrm{d}x_1 \, p(x, t|x_1, t_1) p(x_1, t_1|x_0, t_0).$$

This effectively factorizes the transition probability in time. Work with an intermediate step such that $t_1 = t - \tau$, where τ is small. Show that the transition probability follows the differential equation

$$\frac{\partial}{\partial t} p(x, t|x_0, t_0) = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x, t) p(x, t|x_0, t_0),$$

where the coefficients

$$D^{(n)}(x, t) := \lim_{\tau \to 0} \frac{\mu_n(t|x, t - \tau)}{n! \tau}$$

are called the Kramers–Moyal coefficients.

(c) **Fokker–Planck equation**: Assume that the transition probability $p(x, t|x_0, t - \tau)$ has only two non-zero moments about the mean: $\mu_1(t|x, t - \tau) = \gamma\tau$ and $\mu_2(t|x, t - \tau) = \sigma^2\tau$, where γ and σ are real numbers. What does the differential equation simplify to? (This is called the Fokker–Planck equation—it contains two terms: drift and diffusion.)

4 Random walks (2 points)

Write a script to sample the end-points of two families of random walks:

1. Brownian motion, where steps are drawn according to a standard Gaussian

[2] i.e., does not change with time.

2. A Lévy walk, where steps are drawn according to the Cauchy distribution

$$p(x)\mathrm{d}x = \frac{1}{\pi}\frac{1}{1+x^2}\mathrm{d}x.$$

For both types of random walks, plot the variance of the end points as a function of the number of steps, n, of the walk. Include the theoretical variance for Brownian motion. Use a log-log representation for the plot. Average over 100 random walks each point, and consider the numbers of steps $n = [10, 20, \dots, 1000]$.