Computational Statistics and Data Analysis (MVComp2)

Solutions to exercise 4

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Semester Wi23/24
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1 Exponential distribution (2 points)

The exponential distribution is defined on the interval $[0, +\infty)$ as

$$p(x) \propto \exp(-x).$$

(a) Determine the moment-generating function $m(t)$ for the exponential distribution, $p(x)$.

(b) Verify that $\langle x^n \rangle = n!$.

1.1 Solution

(a) Start from the definition

$$m(t) = E[e^{tX}] = \int_0^\infty \mathrm{d}x e^{(t-1)x} = \left[\frac{e^{(t-1)x}}{t-1}\right]_0^\infty = \frac{1}{1-t}$$

(b) Compute the moments about the origin:

$$\langle x^n \rangle = \int_0^\infty \mathrm{d}x x^n p(x) = \int_0^\infty \mathrm{d}x x^n \mathrm{e}^{-x}$$

Recall the definition of the Gamma function

$$\Gamma(n) = \int_0^\infty \mathrm{d}x x^{n-1} \mathrm{e}^{-x} = (n-1)!,$$

such that $\langle x^n \rangle = \Gamma(n+1) = n!$.

2 Poisson distribution (2 points)

The Poisson distribution $p_\lambda(n)$ is given by

$$p_\lambda(n) = \frac{\lambda^n}{n!}\exp(-\lambda)$$

(a) Find the first three moments about the origin, $\langle n \rangle$, $\langle n^2 \rangle$, and $\langle n^3 \rangle$.[1]

(b) The Poisson distribution has the peculiar property that mean and variance are equal, implying

$$\frac{\langle n^2 \rangle}{\langle n \rangle^2} - 1 = \frac{1}{\langle n \rangle}.$$

Show that this is valid.

2.1 Solution

(a) Use the MGF to compute the first three moments about the origin

$$\langle n^1 \rangle = \left.\frac{\mathrm{d}m(t)}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} = \lambda\mathrm{e}^t\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\Big|_{t=0} = \lambda\mathrm{e}^0\mathrm{e}^{1-1} = \lambda$$

$$\langle n^2 \rangle = \left.\frac{\mathrm{d}^2m(t)}{\mathrm{d}t^2}\right|_{t=0} = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda\mathrm{e}^t\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} = \lambda\mathrm{e}^t\mathrm{e}^{\lambda(\mathrm{e}^t-1)} + \lambda^2\mathrm{e}^{2t}\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\Big|_{t=0} = \lambda + \lambda^2$$

$$\begin{aligned}
\langle n^3 \rangle &= \left.\frac{\mathrm{d}^3m(t)}{\mathrm{d}t^3}\right|_{t=0} \\
&= \left.\frac{\mathrm{d}^3}{\mathrm{d}t^3}\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} \\
&= \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda\mathrm{e}^t\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda\mathrm{e}^t\mathrm{e}^{\lambda(\mathrm{e}^t-1)} + \lambda^2\mathrm{e}^{2t}\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda\mathrm{e}^t\mathrm{e}^{\lambda(\mathrm{e}^t-1)} + 3(\lambda\mathrm{e}^t)^2\mathrm{e}^{\lambda(\mathrm{e}^t-1)} + (\lambda\mathrm{e}^t)^3\mathrm{e}^{\lambda(\mathrm{e}^t-1)}\right|_{t=0} \\
&= \lambda + 3\lambda^2 + \lambda^3
\end{aligned}$$

(b) From (a), we have $\langle n \rangle = \lambda$ and $\langle n^2 \rangle = \lambda + \lambda^2$. From these, the identity is straightforward to verify

$$\frac{\langle n^2 \rangle}{\langle n \rangle^2} - 1 = \frac{\lambda + \lambda^2}{\lambda^2} - 1 = \frac{\lambda}{\lambda^2} = \frac{1}{\langle n \rangle}.$$

[1] A genuine attempt at solving the original assignment will also get you full points.

3 Generalized Fokker–Planck equation (4 points)

We propose to describe a continuous stochastic process of a probability density, where we will make use of a central-moment expansion and assume a Markovian (i.e., memoryless) description.

(a) **Formal-moment expansion**: Consider the (stationary[2]) probability distribution, $p(x)$, given its set of moments about the origin, μ'_n. Show that you can express the distribution in terms of its moments:

$$p(x) = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{\mu'_n}{n!}\delta(x).$$

Hint: The derivatives of the Dirac delta function follow the relation:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,(ik)^n \mathrm{e}^{-ikx} = \left(-\frac{\partial}{\partial x}\right)^n \delta(x)$$

(b) **Kramers–Moyal expansion**: We convert p into a *transition probability function*, $p(x,t|x_0,t_0)$, where x and t denote the spatial coordinate and time, respectively. The moments about the origin become moments about the mean, i.e., we work with μ_n instead of μ'_n. Because of the expansion in (a), the time variable may only affect the moments, not the Dirac delta, i.e.,

$$p(x,t|x_0,t_0) = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{\delta(x-x_0)}{n!}\mu_n(t|x_0,t_0).$$

Invoke Markovianity of p by making use of the Chapman–Kolmogorov equation

$$p(x,t|x_0,t_0) = \int \mathrm{d}x_1\, p(x,t|x_1,t_1)p(x_1,t_1|x_0,t_0).$$

This effectively factorizes the transition probability in time. Work with an intermediate step such that $t_1 = t-\tau$, where τ is small. Show that the transition probability follows the differential equation

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x,t)p(x,t|x_0,t_0),$$

where the coefficients

$$D^{(n)}(x,t) := \lim_{\tau\to 0} \frac{\mu_n(t|x,t-\tau)}{n!\tau}$$

are called the Kramers–Moyal coefficients.

(c) **Fokker–Planck equation**: Assume that the transition probability $p(x,t|x_0,t-\tau)$ has only two non-zero moments about the mean: $\mu_1(t|x,t-\tau) = \gamma\tau$ and $\mu_2(t|x,t-\tau) = \sigma^2\tau$, where γ and σ are real numbers. What does the differential equation simplify to? (This is called the Fokker–Planck equation—it contains two terms: drift and diffusion.)

[2] i.e., does not change with time.

3.1 Solution

(a) Moments about the origin take a simple form

$$\mu'_n = \int \mathrm{d}x\, x^n p(x).$$

Moreover, we can compute the Fourier transform of p

$$\tilde{p}(k) = \int \mathrm{d}x\, \mathrm{e}^{ikx} p(x) = \int \mathrm{d}x \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} x^n p(x) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu'_n.$$

We can plug this in the inverse Fourier transform

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \mathrm{e}^{-ikx} \tilde{p}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \mathrm{e}^{-ikx} \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu'_n = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{\mu'_n}{n!} \delta(x).$$

(b) Use Chapman–Kolmogorov for intermediate step $(x_1, t_1 = t - \tau)$, where τ is small:

$$\begin{aligned} p(x, t|x_0, t_0) &= \int \mathrm{d}x_1 \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{\delta(x - x_1)}{n!} \mu_n(t|x_1, t - \tau) p(x_1, t - \tau|x_0, t_0) \\ &= \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{1}{n!} \mu_n(t|x, t - \tau) p(x, t - \tau|x_0, t_0) \end{aligned}$$

Note that the term $n = 0$ is associated with $\mu_0 = \int \mathrm{d}x\, p(x) = 1$ by normalization. We subtract it out and divide both sides by the lag time, τ

$$\frac{1}{\tau} [p(x, t|x_0, t_0) - p(x, t - \tau|x_0, t_0)] = \frac{1}{\tau} \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{1}{n!} \mu_n(t|x, t - \tau) p(x, t - \tau|x_0, t_0)$$

Take the limit $\tau \to 0^+$ and use the definition of the Kramers–Moyal coefficients to yield the desired differential equation

$$\frac{\partial}{\partial t} p(x, t|x_0, t_0) = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x, t) p(x, t|x_0, t_0),$$

which is called the generalized Fokker–Planck equation.

(c) If only the first two moments are non-zero, then we have

$$\begin{aligned} \frac{\partial}{\partial t} p(x, t|x_0, t_0) &= \left[-\frac{\partial}{\partial x} D^{(1)}(x, t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x, t)\right] p(x, t|x_0, t_0) \\ &= \left[-\gamma \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\right] p(x, t|x_0, t_0). \end{aligned}$$

4 Random walks (2 points)

Write a script to sample the end-points of two families of random walks:

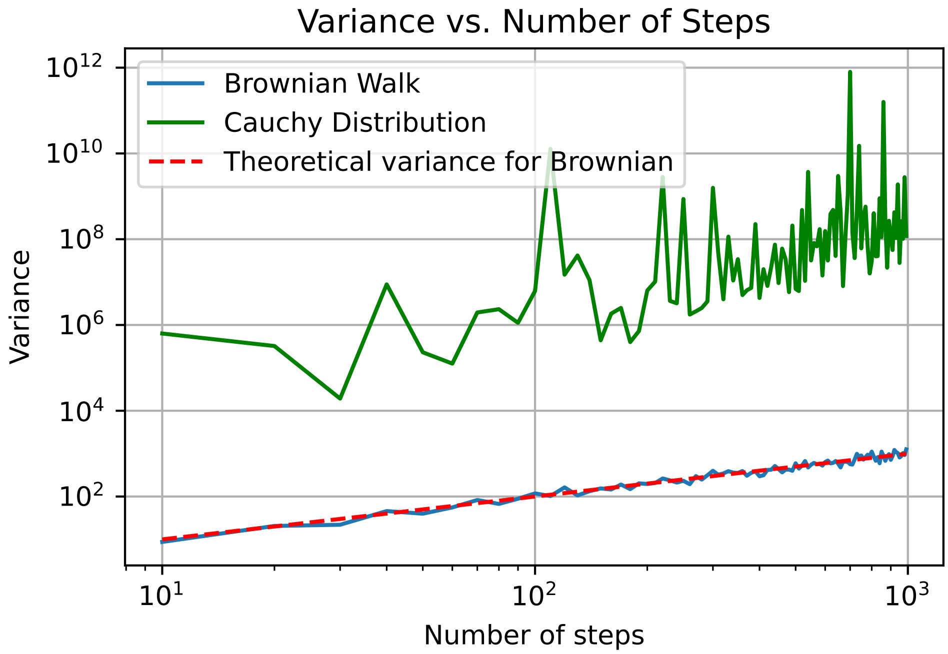
1. Brownian motion, where steps are drawn according to a standard Gaussian
2. A Lévy walk, where steps are drawn according to the Cauchy distribution

$$p(x)\mathrm{d}x = \frac{1}{\pi} \frac{1}{1 + x^2} \mathrm{d}x.$$

For both types of random walks, plot the variance of the end points as a function of the number of steps, n, of the walk. Include the theoretical variance for Brownian motion. Use a log-log representation for the plot. Average over 100 random walks each point, and consider the numbers of steps $n = [10, 20, \dots, 1000]$.

4.1 Solution

```
import numpy as np
import matplotlib.pyplot as plt

def brownian_walk(n) -> float:
    steps = np.random.normal(0, 1, size=n)
    return np.cumsum(steps)[-1]

def cauchy_sample(n) -> float:
    steps = np.sum(np.random.standard_cauchy(n))
    return np.cumsum(steps)[-1]

# Parameters
num_steps_list = np.arange(10, 1000, 10)  # Different numbers of steps to monitor
num_samples = 100  # Number of random walks to sample for each number of steps

variances_brownian = []
variances_cauchy = []

# Compute the variance of the endpoints for each number of steps
for n in num_steps_list:
    endpoints_brownian = [brownian_walk(n) for _ in range(num_samples)]
    variance_brownian = np.var(endpoints_brownian)
    variances_brownian.append(variance_brownian)

    endpoints_cauchy = [cauchy_sample(n) for _ in range(num_samples)]
    variance_cauchy = np.var(endpoints_cauchy)
    variances_cauchy.append(variance_cauchy)

# Plot
plt.plot(num_steps_list, variances_brownian, label='Brownian Walk')
plt.plot(num_steps_list, variances_cauchy, label='Cauchy Distribution', color='green')
plt.plot(num_steps_list, num_steps_list, 'r--', label='Theoretical variance for Brownian')
plt.xlabel('Number of steps')
plt.ylabel('Variance')
plt.legend()
plt.loglog()
plt.title('Variance vs. Number of Steps')
plt.grid(True)
plt.show()
```


Variance vs. Number of Steps
Brownian Walk
Cauchy Distribution
Theoretical variance for Brownian
Variance
Number of steps
10^{12}
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{1}
10^{2}
10^{3}